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HEAT TRANSFER IN COMPLEX-PROFILE CHANNELS UNDER DIFFERENT STABILIZED FLOW CONDITIONS OF THE MEDIUM AND SYSTEM SEARCH FOR HIGHER-ACCURACY NONSTATIONARY TEMPERATURE FIELDS

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By means of the double Laplace–Carson transform as integral averaging of the time function decreasing by the exponential law with weight and along a semi-bounded pipe, the nonstationary heat transfer equation under steady-state laminar or turbulent flow conditions is transformed into a boundary-value problem, which is solved by the method of orthogonal projection of the residual, where, as a finite element, the entire bounded domain of variation of elliptic coordinates is taken.

The complexity of mathematical models of heat transfer processes and new phenomena involving energy transfer associated with the solution of the heat conduction equation calls for the development of more advanced and effective methods for calculating the boundary-value problems of thermal physics. The search for such solution methods is unthinkable without studying the already known methods, which were proposed by researchers at different times, and without comprehending this scientific heritage at the level of the system approach to cognition. In investigating the boundary-value problems of thermal physics, the main sought quantity is temperature which, in the general case, as a function, depends, apart from the input physical parameters, on the coordinates of the current point M(x, y, z) and the time t. In sequential analysis of input given quantities aimed at determining the sought quantity by these arguments, the classification of these equations into parabolic, hyperbolic, and elliptic ones [1] turned out to be useful. Their terminology was taken by analogy with the definitions of equations in partial derivatives of mathematical physics. For example, in the energy transfer equation at nonstationary heat transfer with turbulent ($\varepsilon \neq 0$) or laminar ($\varepsilon = 0$) flow conditions in channels of two-dimensional cross-sections $\Omega(x,\ y - \Omega)$ symmetric about the $0x$ axis, the elliptic coordinates are ξ, η and the unilateral parabolic variables will be X, Fo [2]:

$$\frac{\partial T}{\partial \mathrm{Fo}} + w(\xi, \eta)\frac{\partial T}{\partial X} = \frac{\partial}{\partial \xi}\left((1 + \varepsilon(\xi, \eta))\frac{\partial T}{\partial \xi}\right) + \beta\frac{\partial T}{\partial \eta}\left((1 + \varepsilon(\xi, \eta))\frac{\partial T}{\partial \eta}\right) + \frac{q_v h^2}{\lambda}\Psi_0(\xi, \eta) f(X, \mathrm{Fo}), \tag{1}$$

where

$$\xi = \frac{x}{h};\ \eta = \frac{y}{b};\quad X = \frac{1}{\mathrm{Pe}}\frac{z}{h};\quad \mathrm{Fo} = \frac{at}{h^2};\quad \mathrm{Pe} = \frac{w_0 h}{a};\quad \varepsilon = \frac{\lambda_T}{\lambda};\quad a = \frac{\lambda}{c\gamma};\quad \beta = \frac{h^2}{b^2}.$$

At $w = 0$, $\varepsilon = 0$, $f(X, \mathrm{Fo}) = f(\mathrm{Fo})$ Eq. (1) goes over into the heat conduction equation for long prismatic (cylindrical) bodies, and the replacement of η by $\sqrt{\eta^2 + \zeta^2}$, $\xi = z/b$ leads to the expression

$$\frac{\partial T}{\partial \mathrm{Fo}} = \frac{\partial^2 T}{\partial \xi^2} + \beta\left(\frac{\partial^2 T}{\partial \eta^2} + \frac{\partial^2 T}{\partial \zeta^2}\right) + \frac{q_v h^2}{\lambda}\Psi_0\left(\xi, \sqrt{\eta^2 + \zeta^2}\right) f(\mathrm{Fo}), \tag{2}$$

whose solution represents the temperature fields in axisymmetric bodies. Solutions of Eqs. (1), (2) by exact methods of mathematical physics require knowledge of the spectral problem of the second-order differential operator along el-

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liptic coordinates ξ, η in order that with the help of the found system of eigenfunctions one can form the bases of the rigorous nonalternative space in which analysis of all given input functions of thermal loads is carried out for determining the sought temperature. As is known, such an approach was first used by Fourier in the problem on the cooling of a plate with initial temperature T_0 at a constant temperature T_w on the surfaces, where for determining the temperature field by the method of separation of variables analysis of the only given quantity $T_0 - T_w$ = const was performed by expansion into a series in terms of the system of trigonometric sine functions. The main problem in realizing the Fourier method of separation of variables in problems of mathematical physics is the determination of the base coordinates of the rigorous nonalternative space by solving the Sturm–Liouville problem. Along these lines, the best results have been obtained for one-dimensional regions Ω of simple geometry (plate, cylinder, sphere, cylindrical and spherical shells), for which the systems of eigenfunctions are given by trigonometric and Bessel functions. In determining the eigenfunctions for the second-order self-adjoint differential operator, an important role is played by the hypergeometric Gauss power series [3]. For example, by means of this series the orthogonal bases of rigorous spaces for solving Gretz–Nusselt problems and heat and mass transfer problems in running-down solution layers have been determined [4]. In general, note that almost all special functions of mathematical physics, including the orthogonal Legendre, Laguerre, Chebyshev, and Jacobi polynomials, are somehow associated with the Gauss series [3].

An important achievement in the development of solution methods for problems of mathematical physics was the application of integral transforms of elliptic variables. Despite the fact that the kernels of integral transforms are defined by the eigenfunctions of the problem stated, the application algorithm becomes simpler and more standard than in the Fourier method of separation of variables and makes it possible to find solutions at any time variables of external boundary and internal source loads. In the method of integral transforms, a transition to the region of images of all given input functions takes place as a result of the integral averaging with a special kernel bound by the statement of the problem, and a transformation of the initial differential equation in partial derivatives occurs. Then the image of the sought temperature as a solution of the obtained ordinary first-order differential equation for the time or the unilateral parabolic variable X is defined and a reverse transition to the region of the originals is made. Since the kernels of transforms in finite intervals are defined by discrete spectra (eigenvalues), the temperature field synthesis is given in the form of an infinite series of eigenfunctions, as is customary for the method of separation of variables. In the cases of semibounded and unbounded intervals, the kernels of integral transforms are defined already by continuous spectra, and, therefore, the transition to the original is made by summing over such a spectrum and the solution acquires an integral representation. Systematic use of unilateral and bilateral Fourier transforms in multidimensional regions for investigating heat conduction problems and interrelated heat and mass transfer problems has been made in [5].

In the investigations of the boundary-value problems by the methods of finite-difference approximation (1), (2), the analysis is the introduction into the obtained algebraic system of equations of numerical values of the given functions of coefficient, boundary, and internal thermal loads at discrete points, and the synthesis will be any resolving algorithm of temperature determination in internal nodes of the split domain of variation of elliptic coordinates ξ, η and unilateral variables X, Fo. With boundary conditions of the first kind the error of solution calculation along the direction of variation of the elliptic coordinate increases from zero at one end to a certain value in the middle part, and then decreases to zero at the other end of the interval. Along the direction of variation of unilateral variables the calculation error can accumulate with departure from the beginning and reach an unacceptable large value. A numerical experiment of revealing such a property was considered in [6].

Theoretical investigations of solving Cauchy problems by numerical methods are described in a fundamental monograph [7].

The numerical experiment performed confirms the fact that in developing solution methods for boundary-value problems of Eqs. (1), (2) it is more expedient to use approximate numerical or analytical solutions for coordinates ξ, η, and for unilateral variables the resolving algorithm should be realized by the exact methods of mathematical physics. Among such promising and most effective methods is the numerical-analytical method based on the simultaneous use of the double integral Laplace transform for variables X, Fo and the orthogonal projection of the residual in the domain of variability of elliptic coordinates ξ, η. This method was first proposed in [5].

Note that in developing methods of mathematical physics, beginning with the Fourier method of separation of variables and up to the investigation of the properties of finite-difference schemes, by virtue of the simplicity and graphicness of solutions, the problems of heat conduction in a plate were considered. According to this tradition, let us give the results of calculating the temperature inside a plate at concrete initial and internal source thermal loads by the proposed methods in relative variables Fo = at/l^2, $\xi = x/l$ at the initial conditions

$$T(\xi, 0) = f_0(\xi) = 2T_0 \begin{cases} \xi, & 0 \le \xi \le 0.5 ; \\ 1 - \xi, & 0.5 \le \xi \le 1 . \end{cases}$$

By means of the Laplace integral transform for N = 2, 4, 6 (N is the finite subdivision number at a uniform mesh width), the temperature changes in internal nodes from which the sequence of refinement of the temperature in the center of the plate is written in the form

$$T_1(\text{Fo}) = T_0 \exp(-8\text{Fo}) , \quad T_2(\text{Fo}) = T_0 [1.207 \exp(-9.333\text{Fo}) - 0.207 \exp(-54.627\text{Fo})] ,$$

$$T_3(\text{Fo}) = T_0 [1.244 \exp(-9.648\text{Fo}) - 0.333 \exp(-72\text{Fo}) + 0.089 \exp(-134.352\text{Fo})]$$

have been found. Here, n = 1, 2, 3 ($n = N/2$) is the order of approximation. The procedure of defining the problem solution by the projection method at two initial conditions

$$T(\xi, 0) = f_0(\xi) = T_0 \sin \pi\xi , \quad T(\xi, 0) = f_0(\xi) = 4T_0 (1 - \xi) \xi$$

leads to the temperature fields

$$T(\xi, \text{Fo}) = T_0 \sin \pi\xi \exp(-9.8696\text{Fo}) , \quad T(\xi, \text{Fo}) = 4T_0 (1 - \xi) \xi \exp(-10\text{Fo}) , \tag{3}$$

where the first expression is the exact solution, and the second one describes the temperature homogenization with a high accuracy with a stabilization rate coefficient 10 instead of the exact value π^2 = 9.8696. Note that for the problem in which a numerical experiment has been performed [6], it is impossible to find a solution representation in the form of the second formula of (3), since the initial temperature distribution is given by a nondifferentiable function.

For a plate ($-1 \le \xi = x/R \le 1$) with an internal heat source $\frac{q_v R^2}{\lambda}(1 + \delta\xi^2) f(\text{Fo})$ under the conditions $T(\pm 1, \text{Fo}) = T(\xi, 0) = T_0$ the solution definition in a variety of alternative Riemann spaces along the first optimal base axis in the form

$$\overline{T}(\xi, p) = \frac{T_0}{p} + \overline{a}_1(p) \left[(6 + \delta) - 6\xi^2 - \delta\xi^4 \right]$$

leads to the temperature

$$\Theta(\xi, \text{Fo}) = \frac{[T(\xi, \text{Fo}) - T_0] \lambda}{q_v R^2} = \left[(6 + \delta) - 6\xi^2 - \delta\xi^4 \right] p_1^{(1)}(\delta) \int_0^{\text{Fo}} f(\tau) \exp\left[-p_1^{(1)}(\delta) (\text{Fo} - \tau) \right] d\tau ,$$

where

$$p_1^{(1)}(\delta) = \frac{9 (5\delta^2 + 42\delta + 105)}{14\delta^2 + 144\delta + 378} .$$

Such an integral representation of the temperature defines in the set of control functions $O(f(\text{Fo})) = 1$ higher-accuracy solutions by one component. These solutions satisfy the initial and boundary conditions of the problem, and after the time interval of the transient regime they coincide with the exact values. Small deviations from the exact solution take place only in the middle part of the time interval of the transient regime. For example, at a uniform distribution of the sources (δ = 0) and two control functions $f(\text{Fo}) = 1$, $f(\text{Fo}) = 1 - \exp(-\text{PdFo})$, from this solution we find

$$T(\xi, \mathrm{Fo}) = T_0 + \frac{q_v R^2}{2\lambda}(1-\xi^2)\left[1-\exp(-2.5\mathrm{Fo})\right], \tag{4}$$

$$T(\xi, \mathrm{Fo}) = T_0 + \frac{q_v R^2}{2\lambda}\left[\frac{\mathrm{Pd}\exp(-2.5\mathrm{Fo}) - 2.5\exp(-\mathrm{Pd\,Fo})}{\mathrm{Pd}-2.5}\right](1-\xi^2), \tag{5}$$

which practically coincide with the exact solutions at small deviations in the middle part of the time interval $0 \le \mathrm{Fo} \le 1$. The quantity $p_1^{(1)}(\delta)$ defines the coefficient of the rate of exponential stabilization of the temperature and $p_1^{(1)}(0) = 2.5$; $p_1^{(1)}(-1) = 2.4674$; $p_1^{(1)}(1) = 2.5522$; $p_1^{(1)}(2) = 2.6053$ exceed the exact eigenvalue $\mu_1^2 = \pi^2/4 = 2.4674$. The marked increase in $p_1^{(1)}(\delta)$ with increasing parameter δ means that at such a maximal quadratic distribution of sources in the wall layer the heat is removed into the environment faster. At a special parameter $\delta = -3$, which is found from the condition

$$\int_0^1 \psi_0(\xi)\, d\xi = \int_0^1 (1+\delta\xi^2)\, d\xi = 0,$$

the temperature field for the case of $f(\mathrm{Fo}) = 1$ is reduced to the form

$$T(\xi, \mathrm{Fo}) = T_0 + \frac{q_v R^2}{4\lambda}\left(1-2\xi^2+\xi^4\right)\left[1-\exp(-3\mathrm{Fo})\right]. \tag{6}$$

This temperature field at which the optimal basis satisfies simultaneously the boundary conditions of the first, third, and second kind describes the process of transition to a steady thermal state by a special alternating distribution of the internal thermal load because of the termination of the heat exchange with the environment at adiabatic walls of the plate. Despite the fact that the problems were initially stated at boundary conditions of the first or third kind, the internal thermal loads with distribution functions, the integral of which throughout the body region goes to zero, lead to the simulation of the process of heat conduction at zero boundary conditions of the second kind with termination of the heat removal into the environment. Such thermal states in a plate, a cylinder, a sphere, and other bounded bodies have been considered in [8].

The integral temperature transform $T(x, t)$ with a kernel $R(p, t) = \exp(-pt)$ as a mean-integral averaging of a nonstationary quantity in the time interval t_1

$$\langle T(x, p, t_1)\rangle = \frac{\int_0^{t_1} T(x,t)\, R(p,t)\, dt}{\int_0^{t_1} R(p,t)\, dt} = \frac{\int_0^{t_1} T(x,t)\exp(-p,t)\, dt}{\frac{1}{p}(1-\exp(-p, t_1))}$$

when $t_1 \to \infty$, takes on the form of the equality

$$\lim_{t_1\to\infty} \langle T(x, p, t_1)\rangle = p\int_0^{\infty} T(x,t)\exp(-p,t)\, dt = p\overline{T}(x, p).$$

Thus, the product of the Laplace transform by the parameter p leads to an integral averaging of a nonstationary quantity with an exponentially decreasing kernel $\exp(-pt)$, $\mathrm{Re}p > 0$ throughout the interval of temperature variation t $(0 \le t \le \infty)$, and the two formulas

$$\lim_{t \to 0} T(x, t) = \lim_{p \to \infty} p\overline{T}(x, p) , \tag{7}$$

$$\lim_{t \to \infty} T(x, t) = \lim_{p \to 0} p\overline{T}(x, p) \tag{8}$$

receive concrete thermophysical interpretations. On the basis of the limiting property (7), separating the oscillating part of the image $\overline{T}(x, p)$ in the vicinity of a point at infinity p, A. V. Luikov [9] obtained solutions for the temperature calculation at small Fo numbers and gave an example of the temperature calculation at the moment Fo = 0.0003 by means of 36 terms of the exact solution (Peschl) with the replacement by two terms.

The realization of the orthogonal projection of the residual, when the entire region Ω is taken as a finite element, enables one, in choosing alternative base coordinates and representations of the solution, to use to the full the cross-section geometry of a channel or a prismatic body. In so doing, the limiting property (8) makes it possible to find an optimum coordinate function along whose axis a higher-accuracy temperature field is defined in the form of a single-component representation. Below, on the basis of the application to (1) of the resolving algorithm at the boundary conditions

$$[T(\xi, \eta, X, \mathrm{Fo})]_{X=0} = \varphi_0(\mathrm{Fo}) , \quad [T]_{\mathrm{Fo}=0} = T_0 , \quad [T(\xi, \eta, X, \mathrm{Fo})]_{\Gamma} = \varphi(\xi, X, \mathrm{Fo}) \tag{9}$$

we give a brief review of the system approach to the development of methods for investigating internal problems of heat exchange in direct channels and nonstationary heat conduction in multidimensional bodies. The preference for solving boundary-value problems with boundary conditions of the first kind is justified by the fact that in experimental studies of physical quantities by more advanced and exact methods success has been achieved in the technique of temperature measurement at one point or over the entire surface of a body. Therefore, from the boundary temperature measurement data as a yield of the intermediate response of the thermal load ad its interpolation by the function $\varphi(\xi, X, \mathrm{Fo})$ in the boundary conditions (9), we can determine theoretically the temperature field by solving the boundary-value problem. This makes it possible to investigate the mechanism of heat exchange of a prismatic body or a fluid flow in a two-dimensional channel with a transversely incoming external medium, i.e., enables us to propose a simplified model for solving a complex problem of conjugate heat exchange between the body and the medium flowing past it.

Instead of using sequentially the integral Laplace transform for the variables X and Fo, let us consider the double Laplace–Carson transform [10]:

$$\overline{T}^*(\xi, \eta, s, p) = sp \int_0^\infty \int_0^\infty T(\xi, \eta, X, \mathrm{Fo}) \exp[-(sX + p\mathrm{Fo})]\, dX d\mathrm{Fo} .$$

Then for $\overline{T}^*(\xi, \eta, s, p)$ the boundary-value problem (1), (9) is reduced to the form

$$L\left[\overline{T}^*(\xi, \eta, s, p)\right] = -\left[\frac{\partial}{\partial \xi}\left((1 + \varepsilon)\frac{\partial \overline{T}^*}{\partial \xi}\right) + \beta\frac{\partial}{\partial \xi}\left((1 + \varepsilon)\frac{\partial \overline{T}^*}{\partial \eta}\right)\right]$$

$$+ (p + sw)\overline{T}^*(\xi, \eta, s, p) = \frac{q_v h^2}{\lambda}\psi_0(\xi, \eta)\overline{f}^*(s, p) + (T_0 p + sw\overline{\varphi}_0) , \tag{10}$$

$$\left[\overline{T}^*(\xi, \eta, s, p)\right]_{\Gamma} = \overline{\varphi}^*(\xi, s, p) \tag{11}$$

and its solution can be given as

$$\overline{T}^* (\xi, \eta, s, p) = \overline{\varphi}^* (\xi, s, p) + \sum_{k=1}^{n} \overline{a}_k^* (s, p)\, \psi_k (\xi, \eta)\,, \tag{12}$$

where the choice of coordinate functions $\psi_k(\xi, \eta)$ of the alternative space is limited by only one requirement $[\psi_k]_\Gamma = 0$ needed for making (12) consistent with the boundary condition (11). The coefficients $\overline{a}_k^*$ (s, p) are determined from the requirement that the residual

$$\varepsilon_n \left[\overline{a}_1^*, ..., \overline{a}_n^*, \xi, \eta\right] = L\left[\overline{T}_n^*\right] - \left[T_0 p + sw (\xi, \eta)\, \overline{\varphi}_0 (p)\right] - \frac{q_v h^2}{\lambda}\, \psi_0 (\xi, \eta)\, \overline{f}^* \neq 0$$

be orthogonal to the coordinate functions $\psi_j(\xi, \eta)$ throughout the region of the clear section of the channel, i.e., from the condition

$$\int_\Omega \varepsilon_n \left[\overline{a}_1^* (s, p), ..., \overline{a}_n^* (s, p), \xi, \eta\right] \psi_j (\xi, \eta)\, d\xi d\eta = 0\,, \quad j = 1, 2, ..., n\,.$$

This system in matrix writing has the form

$$(A + sB + pC)\, \overline{a}^* (s, p) = \left[T_0 - \overline{\varphi}^* (s, p)\right] pD + \left[\overline{\varphi}_0 (p) - \overline{\varphi}^* (s, p)\right] sF + \frac{q_v h^2}{\lambda} \overline{f}^* E\,, \tag{13}$$

where, without loss of generality of the method, $\varphi(\xi, X, \text{Fo}) = \varphi(X, \text{Fo})$ is assumed and matrix elements are calculated by the formulas

$$A_{jk} = -\int \left[\frac{\partial}{\partial \xi}\left((1 + \varepsilon)\, \frac{\partial \psi_k}{\partial \xi}\right) + \beta\, \frac{\partial}{\partial \eta}\left((1 + \varepsilon)\, \frac{\partial \psi_k}{\partial \eta}\right)\right] \psi_j d\sigma = \int (1 + \varepsilon) \left(\frac{\partial \psi_k}{\partial \xi}\frac{\partial \psi_j}{\partial \xi} + \frac{\partial \psi_k}{\partial \eta}\frac{\partial \psi_j}{\partial \eta}\right) d\sigma = A_{kj} > 0\,,$$

$$B_{jk} = \int w \psi_k \psi_j d\sigma\,, \quad C_{jk} = \int \psi_k \psi_j d\sigma\,, \quad D_j = \int \psi_j d\sigma\,, \quad F_j = \int w \psi_j d\sigma\,, \quad E_j = \int \psi_0 \psi_j d\sigma\,, \tag{14}$$

$$d\sigma = d\xi d\eta\,.$$

In Laplace transforms, the parameters p and s are such that $p + sw(\xi, \eta) > 0$, i.e., the operator $L[\overline{T}_n^*\,]$ is positive and self-adjoint. Therefore, in the discrete analog the matrixes A, B, C approximating this continuous differential operator are symmetric with positive elements. Consequently, the roots of the algebraic equations $\Delta(p) = \left|A + pC\right| = 0$, $\Delta(s) = \left|A + sB\right| = 0$ will be simple and negative. Let us denote them in increasing order of the absolute values as

$$-p_1^{(n)} < 0\,, \quad -p_2^{(2)} < 0\,, ..., -p_n^{(n)} < 0\,; \quad -s_1^{(n)} < 0\,, ..., -s_n^{(n)} < 0 \quad \left(p_k^{(n)} > 0\,, \quad s_k^{(n)} > 0\right).$$

Assume in Eq. (1) $\partial T/\partial \text{Fo} = 0$ and in the boundary conditions (9)

$$\varphi (X, \text{Fo}) = \varphi (X)\,, \quad \varphi_0 (\text{Fo}) = T_0\,, \quad f (X, \text{Fo}) = f (X)\,,$$

Then for the stationary heat transfer in the Laplace transforms

$$T^* (\xi, \eta, s) = \varphi^* (s) + \sum_{k=1}^{n} a_k^* (s)\, \psi_k (\xi, \eta) \tag{15}$$

from the constitutive equation (13) at $p = 0$ the elements of the response matrix $a^*(s)$, according to the Cramer formula, are equal to

$$a_k^*(s) = \frac{[T_0 - s\varphi^*(s)]\,\Delta_k(s, F)}{\Delta(s)} + \frac{q_v h^2}{\lambda}\frac{f^*(s)\,\Delta_k(s, E)}{\Delta(s)}, \tag{16}$$

where $\Delta_k(s, N) = \sum_{j=1}^{n} N_j \Delta_{jk}(s)$, $\Delta_{jk}(s)$ are cofactors of the main determinant $\Delta(s) = |A + sB|$. The transfer functions $\Delta_k(s, F)/\Delta(s)$, $\Delta_k(s, E)/\Delta(s)$ are proper algebraic fractions, and decomposing them in terms of the simple poles of the denominator into sums of blocks of responses of elementary thermoinertial units to the loads $T_0 - s\varphi^*(s)$, $f^*(s)$, we get

$$a_k^*(s) = \sum_{j=1}^{n} \frac{\Delta_k\left(-s_k^{(n)}, F\right)}{\Delta'\left(-s_i^{(n)}\right)} \frac{\left[T_0 - s\varphi^*(s)\right]}{s + s_i^{(n)}} + \frac{q_v h^2}{\lambda} \sum_{i=1}^{n} \frac{\Delta_k\left(-s_i^{(n)}, E\right)}{\Delta'\left(-s_i^{(n)}\right)} \frac{f^*(s)}{s + s_i^{(n)}}, \quad \Delta' = \frac{d\Delta}{ds}. \tag{17}$$

This formula makes it possible to determine, by a single-type reverse transition in each block, $a_k(X)$ at particular values of $\varphi^*(s)$, $f^*(s)$ and write the temperature $T(\xi, \eta, X)$ in the fluid flow inside a channel with a two-dimensional cross-section in terms of representation (15). In the general case, at $\varphi(0) = T_0$ by the transition formula $\varphi'(X) \Leftrightarrow s\varphi^*(s) - T_0$ and the convolution theorem we obtain

$$T_n(\xi, \eta, X) = \varphi(X) - \sum_{i=1}^{n} \int_0^X \frac{d\varphi}{d\alpha} \exp\left[-s_i^{(n)}(X - \alpha)\right] d\alpha\, \psi_i^{(n)}(\xi, \eta, F)$$

$$+ \frac{q_v h^2}{\lambda} \sum_{i=1}^{n} \int_0^X f(\alpha) \exp\left[-s_i^{(n)}(X - \alpha)\right] d\alpha\, \psi_i^{(n)}(\xi, \eta, E), \tag{18}$$

where

$$\psi_i^{(n)}(\xi, \eta, N) = \sum_{k=1}^{n} \frac{\Delta_k\left(-s_k^{(n)}, N\right)}{\Delta'\left(-s_i^{(n)}\right)} \psi_k(\xi, \eta).$$

Assume in Eq. (1) $w(\xi, \eta) = 0$ and in the thermal loads $f(X, \mathrm{Fo}) = f(\mathrm{Fo})$, $\varphi(X, \mathrm{Fo}) = \varphi(\mathrm{Fo})$, $\varphi_0(\mathrm{Fo}) = T_0$. Then the nonstationary temperature field in a prismatic (cylindrical) rod is given by the analogous formula

$$T_n(\xi, \eta, \mathrm{Fo}) = \varphi(\mathrm{Fo}) - \sum_{i=1}^{n} \int_0^{\mathrm{Fo}} \frac{d\varphi}{d\tau} \exp\left[-p_i^{(n)}(\mathrm{Fo} - \tau)\right] d\tau\, \psi_i^{(n)}(\xi, \eta, D)$$

$$+ \frac{q_v h^2}{\lambda} \sum_{i=1}^{n} \int_0^{\mathrm{Fo}} f(\tau) \exp\left[-p_i^{(n)}(\mathrm{Fo} - \tau)\right] d\tau\, \psi_i^{(n)}(\xi, \eta, E). \tag{19}$$

If instead of η in (19) we assume $\sqrt{\eta^2 + \zeta^2}$, then we get the temperature $T_n(\xi, \sqrt{\eta^2 + \zeta^2}, \mathrm{Fo})$ inside an axisymmetric body as a solution of Eq. (2) in the space $\psi_k(\xi, \sqrt{\eta^2 + \zeta^2})$. In so doing, the matrix elements in system (13) are determined by calculating integrals (14) over the three-dimensional region Ω.

The solution for the complete truncated system (13) of the first order in the case of $\varphi_0(\mathrm{Fo}) = T_0$ will be

$$\bar{a}_1^*(s, p) = \frac{(a_1 s + b_1 p)\left(T_0 - \bar{\varphi}^*(s, p)\right)}{p + \gamma_1^{(1)} s + p_1^{(1)}} + \frac{q_v h^2}{\lambda} \frac{e_1 \bar{f}^*(s, p)}{p + \gamma_1^{(1)} s + p_1^{(1)}}, \tag{20}$$

where $a_1 = F_1/C_{11}$; $b_1 = D_1/C_{11}$; $\gamma_1^{(1)} = p_1^{(1)}/s_1^{(1)} = B_{11}/C_{11}$; $p_1^{(1)} = A_{11}/C_{11}$; $s_1^{(1)} = A_{11}/B_{11}$; $e_1 = E_1/C_{11}$.

Representing the nonstationary temperature in the first approximation leads to the consideration of the transfer function of an elementary thermoinertial unit with two parameters, and by the transition formula [10]

$$\frac{(a_1 p + b_{1p})}{p + \gamma_1^{(1)} s + p_1^{(1)}} \Leftrightarrow \begin{Bmatrix} a_1 \exp\left(-p_1^{(1)}\,\mathrm{Fo}\right), & X > \gamma_1^{(1)}\mathrm{Fo} \\ b_1 \exp\left(-s_1^{(1)}\,\mathrm{Fo}\right), & X < \gamma_1^{(1)}\mathrm{Fo} \end{Bmatrix}, \quad \frac{1}{p + \gamma_1^{(1)} s + p_1^{(1)}} = \frac{1}{p_1^{(1)}}\left(1 - \frac{s\gamma_1^{(1)} + p}{p + \gamma_1^{(1)} s + p_1^{(1)}}\right)$$

and the convolution theorem the value of $a_1(X, \mathrm{Fo})$ is determined. The solution is defined by two lines: the upper line describes the change in the temperature of the liquid, which at the instant of Fo = 0 was already inside the channel, and the lower line defines the temperature of the medium that has got into the channel (Fo > 0). In the solutions in the second and subsequent approximations, the upper line is given by formula (19) and the lower one is written as the temperature formula (18) [5].

In choosing base coordinates for investigating boundary-value problems of the first kind by representing solution (12) in a variety of alternative Riemann spaces, of great importance is the blending boundary function of the clear section profile of the channel, i.e., the equation of the region Ω boundary. If we find a function $\omega(\xi, \eta)$ that is greater than zero inside the channel and equal to zero on the wetted surface, then it is more expedient to take for the bases of the function space the system

$$\psi_k(\xi, \eta) = \omega(\xi, \eta)\, \xi^{(k-1)}\, \eta^{2(k-1)}, \quad k = 1, 2, \ldots, n.$$

Methods for composing $\omega(\xi, \eta)$ and choosing an optimal coordinate function depending on the states of the internal or external boundary thermal loads were developed in [11, 12].

On the basis of the system approach to the solution methods for a combined energy transfer equation in a planar channel ($m = 0$, $-1 \le \xi = x/R \le 1$) and a circular pipe ($m = 1.0 \le \xi = r/R \le 1$)

$$\frac{\partial T}{\partial \mathrm{Fo}} + w(\xi, m)\frac{\partial T}{\partial X} = \frac{1}{\xi^m}\frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial T}{\partial \xi}\right) + \frac{q_v h^2}{\lambda}\,\psi_0(\xi)\, f(X, \mathrm{Fo}) \tag{21}$$

at the generalized boundary conditions of the third kind

$$[T(\xi, X, \mathrm{Fo})]_{X=0} = \varphi_0(\mathrm{Fo}), \quad [T(\xi, X, \mathrm{Fo})]_{\mathrm{Fo}=0} = T_0, \quad 0 \le X < \infty,$$
$$\left(\frac{\partial T}{\partial \xi} + \mathrm{Bi}\, T(\xi, X, \mathrm{Fo})\right)_{\xi=1} = \mathrm{Bi}\left(\varphi(X, \mathrm{Fo}) + \frac{q}{\alpha}\right), \quad \left(\frac{\partial T}{\partial \xi}\right)_{\xi=0} = 0 \tag{22}$$

let us compare a series of special problems to the literature solutions and present some new results of investigations. In the formulation of the problem $X = \dfrac{1}{\mathrm{Pe}}\dfrac{z}{R}$, $\mathrm{Pe} = \dfrac{w_0 R}{a}$, $\mathrm{Bi} = \dfrac{\alpha R}{\lambda}$, and $\Phi(X, \mathrm{Fo}) = \varphi + \dfrac{q}{\alpha}$ is the generalized reduced temperature of the environment. It is customary in solar-energy technology to call the quantity $\Phi(X, \mathrm{Fo}) = T_{\mathrm{w}} + \dfrac{q}{\alpha} =$ const, according to Duffy, the generalized air temperature. The solution in Laplace–Carson transforms satisfying exactly the boundary conditions (22) is in the family of the linear composition

$$\overline{T}_n^*(\xi, s, p) = \overline{\Phi}^*(s, p) + \sum_{k=1}^{n} \overline{a}_k^*(s, p)\left(\frac{\mathrm{Bi} + 2k}{\mathrm{Bi}} - \xi^{2k}\right).$$

For steady heat exchange ($\partial T/\partial \mathrm{Fo} = 0$), when $\Phi(X, \mathrm{Fo}) = T_{\mathrm{w}} + \dfrac{q}{\alpha} =$ const, $q_v(\xi, X) = q_v =$ const, the temperature change as a solution of the generalized Gretz–Nusselt problem is reduced to the form

$$\Theta_n(\xi, X, \mathrm{Bi}, m) = \frac{T(\xi, X) - \left(T_{\mathrm{w}} + \dfrac{q}{\alpha}\right)}{T_0 - \left(T_{\mathrm{w}} + \dfrac{q}{\alpha}\right)} = \sum_{k=1}^{n} \psi_k^{(n)}(\xi, \mathrm{Bi}, m, D) \exp\left[- s_k^{(n)}(\mathrm{Bi}, m) X\right]$$

$$+ \mathrm{Po}\left[\frac{1}{2(m+1)}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right) + \sum_{k=1}^{n} \psi_k^{(n)}(\xi, \mathrm{Bi}, m, E) \exp\left[- s_k^{(n)}(\mathrm{Bi}, m) X\right]\right],$$

where $\mathrm{Po} = q_v R^2/[\lambda(T_0 - (T_{\mathrm{w}} + q/\alpha))]$ is the Pomerantsev number. The input values in this solution have been found with a Poiseuille velocity $w(\xi, m) = 0.5(m+3)(1-\xi^2)$ up to the third order of approximation (n = 1, 2, 3) for numbers Bi = 1, 4, 10, ∞. At a fixed order of n the deviation of the approximate solution increases with increasing Bi, and the largest calculation error is obtained in the problem with boundary conditions of the first kind. Even at such properties the values of $\psi_1^{(3)}(\xi, \infty, m, D)$, $s_1^{(3)}(\infty, m)$ practically coincided with the expressions in the first term of the Gretz–Nusselt solution. It is enough to carry out the thermal calculation by the temperature field in the third approximation. Refinement of the solution in the fourth and subsequent approximations is mainly of theoretical interest. For a circular pipe with boundary conditions of the first kind, the following convergence of the eigenvalues has been found: $s_1^{(1)} = 4$, $s_1^{(2)} = 3.671$, $s_2^{(2)} = 36.333$; $s_1^{(3)} = 3.6569$, $s_2^{(3)} = 23.938$, $s_3^{(3)} = 161.20$; $s_1^{(4)} = 3.6568$, $s_2^{(4)} = 22.444$, $s_3^{(4)} = 74.428$, $s_4^{(4)} = 532.27$, i.e., the rates of exponential temperature stabilizations along the length of the channel practically coincide with the exact changes from the second order of approximation.

The local Nusselt numbers obtained by the formulas

$$\mathrm{Nu}(X, m, \mathrm{Bi}) = -\frac{2}{\langle \Theta(X, m, \mathrm{Bi}) \rangle}\left(\frac{\partial \Theta}{\partial \xi}\right)_{\xi=1}, \quad \Theta = \sum_{k=1}^{3} \psi_k^{(3)}(\xi, \mathrm{Bi}, m) \exp\left(- s_k^{(3)} X\right),$$

gave good agreements with the known solutions [2], and the limiting values $\lim_{X\to\infty} \mathrm{Nu}(X, 0, \infty) = 3.77$, $\lim_{X\to\infty} \mathrm{Nu}(X, 1, \infty) = 3.66$ are equal to the exact minimum Nusselt values.

Let us give two examples of higher-accuracy calculations of the temperature fields along one space axis with an optimal coordinate function, which is chosen for individual thermophysical processes and depending on the kind of external boundary or internal source thermal loads. The optimal coordinate function at a uniform distribution of internal sources ($\psi_0(\xi) = 1$) will be $\psi_1(\xi, m) = \dfrac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2$. The combined representation of the solution of the boundary-value problem

$$\frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial T^*}{\partial \xi}\right) - \left[sT^*(\xi, s) - T_0\right] w(\xi, m)\, \xi^m = -\frac{q_v R^2}{\lambda} \xi^m f^*(s),$$

$$\left(\frac{\partial T^*}{\partial \xi} + \mathrm{Bi}\, T^*(\xi, s)\right)_{\xi=1} = \mathrm{Bi}\, T_0/s, \quad \left(\frac{\partial T^*}{\partial \xi}\right)_{\xi=0} = 0$$

leads to the coefficient

$$a_1^*(s, m) = \frac{q_v R^2}{2\lambda(m+1)} \frac{s_1^{(1)}(\mathrm{Bi}, m) f^*(s)}{s + s_1^{(1)}(\mathrm{Bi}, m)},$$

and at an arbitrary control function $f(X)$ the temperature is defined by the formula

$$T(\xi, X, m, \mathrm{Bi}) = T_0 + \frac{q_v R^2}{2\lambda(m+1)}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right) s_1^{(1)}(\mathrm{Bi}, m) \int_0^X f(\alpha) \exp\left[- s_1^{(1)}(X-\alpha)\right] d\alpha ,$$

whence in the fluid flow inside a circular pipe at $m = 1$, $f(X) = 1$ will be written as

$$T(\xi, X, m, \mathrm{Bi}) = T_0 + \frac{q_v R^2}{4\lambda}\left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2\right)\left[1 - \exp\left(-\frac{12\mathrm{Bi}(\mathrm{Bi}+4)X}{3\mathrm{Bi}^2 + 16\mathrm{Bi} + 24}\right)\right] \tag{23}$$

and after the initial part of the pipe this expression coincides with he exact solution.

For the second problem, let us consider the linear change in the ambient temperature ($T_0 + \Delta TX$, $q_v = 0$) or the change that goes over into such a function with increasing X. The optimal coordinate function for the circular pipe will be

$$\psi_1(\xi, \mathrm{Bi}) = \frac{4+3\mathrm{Bi}}{\mathrm{Bi}} - 4\xi^2 + \xi^4 , \tag{24}$$

and the temperature field along this space axis at $\mathrm{Bi} = \infty$ takes on the form

$$T(\xi, X) = T_0 + \Delta TX - \frac{\Delta T}{8}\left(3 - 4\xi^2 + \xi^4\right)(1 - \exp(-3.729X)) . \tag{25}$$

After the initial part of the pipe this solution goes over into a self-similar expression which will coincide with the exact temperature change. We can find a solution with such a property along the axis (24) for the thermal load

$$\varphi(X) = T_0 + \Delta TX(1 - \exp(-\mathrm{Pd}X)) .$$

Refinement of solution (25) in the second approximation in the space $\psi_k = (3 - 4\xi^2 + \xi^4)\xi^{2(k-1)}$ leads to the expression

$$\Theta(\xi, X) = \frac{T(\xi, X) - T_0}{\Delta T} = X - 0.125\left(3 - 4\xi^2 + \xi^4\right)[1 - 1.056\exp(-3.662X)$$

$$+ 0.056\exp(-30.863X)] + 0.0402\left(3\xi^2 - 4\xi^4 + \xi^6\right)[\exp(-3.662X) - \exp(-30.863X)] . \tag{26}$$

From (25) and (26) the heat inflow needed for a linear increase in the wall temperature along the flow is defined, respectively, by the formulas

$$q(X) = -\lambda\left(\frac{\partial T}{\partial r}\right)_{r=R} = \frac{\lambda}{R}\left(\frac{\partial T}{\partial \xi}\right)_{\xi=1} = \frac{\lambda \Delta T}{2R}\left[1 - \exp(-3.729X)\right] .$$

$$q(X) = \frac{\lambda \Delta T}{2R}\left[1 - 0.734\exp(-3.662X) - 0.266\exp(-30.863X)\right],$$

which practically coincide with one another along the full length of the pipe.

In heat-transfer problems, beginning with the classical Gretz–Nusselt statements, mainly velocity values of steady-state isothermal flows were introduced into Eq. (21) despite the fact that the heat transfer is investigated under the conditions of heating or cooling of the fluid flow through the channel walls. If the relative velocity $w(\xi, \eta)$ of the isothermal flow is determined by the solution of the Poisson equation at a constant quantity on the right side $-h^2/\mu\ \partial P/\partial z = \mathrm{const} < 0$, then under the conditions of nonisothermal flow when the stationary temperature field depends only on the coordinates ξ, η, the dynamic viscosity coefficient should be also taken as a variable quantity of these arguments. Methods for defining the approximate analytical expression of velocity as a solution of the Poisson equation at constant and variable fluid moving forces $(h^2/\mu)\,\partial P/\partial z = f(\xi, \eta)$ were developed in [11, 13].

In a circular pipe at $f(\xi,\ \delta)\ =\ \dfrac{R^2(1+\delta\xi^2)}{\mu_0}\dfrac{\partial P}{\partial z}$ the exact solution will be

$$W(\xi,\delta)=\frac{3w_0}{2(3+\delta)}\left[4+\delta-\left(4\xi^2+\delta\xi^4\right)\right],\quad w_0=\frac{R^2(3+\delta)}{24\mu_0}, \tag{27}$$

which at $\delta = 0$ goes over into the Poiseuille distribution. The dynamic viscosity coefficient $\mu(\xi,\ \delta)\ =\ \dfrac{\mu_0}{1+\delta\xi^2}$, where $\mu(0,\ \delta)\ =\ \mu_0,\ \mu(1,\ \delta)\ =\ \dfrac{\mu_0}{1+\delta}$ are the extreme values, $\delta>0$ under heating and $\delta<0$ under cooling of the liquid through the channel walls. The velocity profiles become filled more homogeneously in the case of heating and prolate in the flow core upon cooling. The velocity curve has an inflection point only in the case of cooling ($\delta<0$). Differentiating solution (27), we obtain the expression

$$\left(\frac{\partial W}{\partial\xi}\right)_{\xi=1}=-\frac{2R^2(2+\delta)\,w_0}{3+\delta},$$

taking a zero value with the parameter $\delta = -2$. In the distribution with alternating signs, the moving force of a pumped fluid flow with the special parameter $\delta^* = -2$ at which the friction force disappears on the pipe surface is the root of the equation

$$(m+1)\int_0^1 f(\xi,\delta)\,\xi^m d\xi=0\,,\quad m=0\,,\quad m=1\,,$$

and for two-dimensional convex profiles this condition goes over into equality to zero of the channel's cross-section integral of the variable force $f(\xi,\ \eta)$.

For heat-transfer problems in channels with turbulent flow conditions, the given resolving algorithm makes it possible to attain the aim only at given values of the velocity $w(\xi,\ \eta)$ and the function $\varepsilon(\xi,\ \eta)$ in the energy transfer equation (1). Despite the fact that the function ε enters under the sign of derivatives, in the realization of the method integration over this quantity is only performed by formulas (14). Therefore, small errors introduced by approximate values of $\varepsilon(\xi,\ \eta)$ and $w(\xi,\ \eta)$ lead to small deviations in determining the temperature field. The values of A_{11} and other elements of the matrix A increase with increasing turbulent effect $\varepsilon>0$. For the matrix, B_{11} and other elements decrease at a more filled velocity profile and take minimum values in a plug flow. Consequently, the eigenvalues $s_1^{(n)}$ and $p_1^{(n)}$ in the first approximation are $S_1^{(1)} = A_{11}/B_{11}$, $p_1^{(1)} = A_{11}/C_{11}$ and in the subsequent approximations under turbulent conditions they take larger values than in laminar flows, and this confirms the higher stabilization rates of the temperature fields and the heat transfer intensity in turbulent flows of the medium. Thus, the simplification of the problem for the turbulent flow of a molten metal by considering the rod flow [14] is fully justified by the above derivations.

The isothermal flow velocity in an equilateral triangular channel is expressed exactly in terms of the composite boundary function in the form

$$w(\xi,\eta)=W(\xi,\eta)/w_0=15\left(\xi^2-\eta^2\right)(1-\xi)$$

and in the flow core max $W = W(2/3,\ 0) = 2.222w_0$. The change in the stationary temperature as a solution of Eq. (1) at $\beta = 3$, $\partial T/\partial\mathrm{Fo} = 0$ and at coordinate functions

$$\psi_1(\xi,\eta)=\omega(\xi,\eta)=\left(\xi^2-\eta^2\right)(1-\xi)\,,\quad \psi_2=\omega\eta^2\,,\quad \psi_3=\omega\xi^2\,,$$

as well as at constant boundary conditions ($\varphi(X) = T_\mathrm{w}$, $\psi_0 = 0$) is reduced to the form

$$\Theta_n(\xi, \eta, X) = \frac{T_n(\xi, \eta, X) - T_\mathrm{w}}{T_0 - T_\mathrm{w}} = \omega(\xi, \eta) \sum_{i=1}^{n} \psi_i^{(n)}(\xi, \eta) \exp\left(-s_i^{(n)} X\right), \tag{28}$$

where the values of $s_i^{(n)}$ and $\psi_i^{(n)}$ have been found up to the third order of approximation. The calculations of the eigenvalues yielded $s_1^{(1)} = 25.667$, $s_1^{(2)} = 24.084$, $s_2^{(2)} = 198.812$, $s_1^{(3)} = 24.077$, $s_2^{(3)} = 85.509$, $s_3^{(3)} = 205.479$, and for the eigenfunctions in the third approximation

$$\psi_1^3(\xi, \eta) = 9.738 - 12.348\eta^2 + 0.486\xi^2, \quad \psi_2^{(3)}(\xi, \eta) = 1.545 + 0.073\eta^2 + 3.402\xi^2,$$

$$\psi_3^3(\xi, \eta) = 0.219 + 43.707\eta^2 - 4.754\xi^2.$$

It can easily be shown that at all similar points of the three faces the local thermal flows are identical. Therefore, the changes in the thermal flows q on the walls can be determined by means of the formula for the face $x = h$:

$$q_n^*(\eta, X) = \frac{q(\eta, X)\, h}{\lambda (T_0 - T_\mathrm{w})} = -\left(\frac{\partial \Theta_n}{\partial \xi}\right)_{\xi=1} \tag{29}$$

and by two approximations

$$q_1^*(\eta, X) = 9.167(1 - \eta^2) \exp(-25.667X), \quad q_2^*(\eta, X) = (1 - \eta^2)$$

$$\times \left[(9.526 - 12.475\eta^2) \exp(-24.084X) + (-1.989 + 42.503\eta^2) \exp(-198.812X)\right]. \tag{30}$$

The change in the local Nusselt number is determined by the mass-average temperature

$$\mathrm{Nu}^{(n)}(X) = -\frac{1}{4\langle\Theta(X)\rangle} \frac{d\langle\Theta(X)\rangle}{dX}, \quad \langle\Theta(X)\rangle = \frac{\int \Theta w d\xi d\eta}{\int w d\xi d\eta}, \tag{31}$$

and by the solution in the second approximation in the form

$$\mathrm{Nu}^{(n)}(X) = \frac{2.676[1 + 0.797 \exp(-79.768X)]}{1 + 0.108 \exp(-79.768X)}. \tag{32}$$

Determinations of the values of (29) and (31) by the temperature fields in the third approximations have led to minor refinements of the values of (30) and (32). The $\mathrm{Nu}^{(2)}(X)$, $\mathrm{Nu}^{(3)}(X)$ and q_2^*, q_3^* curves have practically merged:

$$\lim_{x\to\infty} \mathrm{Nu}^{(2)}(X) = 2.6757, \quad \lim_{x\to\infty} \mathrm{Nu}^{(3)}(X) = 2.6753.$$

The theoretical studies with the use of different formulas for calculating Nusselt numbers made for the temperature field (28) in the third approximation gave good agreement with the experimental data obtained by the researchers of the Polzunov Central Boiler-Turbine Institute (TsKTI) and with the results of investigations carried out by I. Cox and R. Stevens, which are presented in [2].

The problem of nonstationary heat transfer with constant boundary conditions of the first kind was investigated by Siegel in [15], where upon integral averaging the equation

$$\frac{\partial}{\partial \mathrm{Fo}} \int_0^1 T(\xi, X, \mathrm{Fo})\, \xi^m d\xi + 0.5(m + 3) \frac{\partial}{\partial X} \int_0^1 \left(1 - \xi^2\right) T(\xi, X, \mathrm{Fo})\, \xi^m d\xi = \int_0^1 \frac{\partial}{\partial \xi}\left(\xi^m \frac{\partial T}{\partial \xi}\right) d\xi \tag{33}$$

is solved by the method of characteristics for variables X, Fo and by the temperature representation for ξ by a power polynomial. However, such an approach leads to worse results than the calculation by the proposed algorithm. Even in the seventh order of approximation the first eigenvalues of $\alpha_0 = 5.1540$, $\beta_0 = 7.3136$ have been found. While β_0 is fairly close to the exact value of $2s_1 = 7.316$, α_0 is 10.9% less than $s_1 = 5.783$. More detailed comparisons were made as long ago as in [5]. Interestingly, equality (33) is essentially an orthogonal projection of the residual at boundary conditions of the second kind along the first space axis, since in such problems for any bounded domain Ω the first eigenfunction is $\psi_1 = 1$ with a value of $\mu_1^2 = 0$. If we make the assumption

$$(m+1)\int_0^1 T(\xi, X, \mathrm{Fo})\,\xi^m d\xi = (m+1)\int_0^1 w(\xi, m)\,T(\xi, X, \mathrm{Fo})\,\xi^m d\xi = T(1, X, \mathrm{Fo}) = T(X, \mathrm{Fo}),$$

which is true only at small Bi numbers (Bi << 1), then the introduction of the value of $\left(\frac{\partial T}{\partial \xi}\right)_{\xi=1}$ into the right side of (33) from the constant boundary conditions (22) leads with respect to $T(X, \mathrm{Fo})$ to the equation

$$\frac{\partial T}{\partial \mathrm{Fo}} + \frac{\partial T}{\partial X} + \mathrm{Bi}\,(m+1)\,T(X, \mathrm{Fo}) = \mathrm{Bi}\,(m+1)\left(T_{\mathrm{w}} + \frac{q}{\alpha}\right)$$

and the solution under the conditions $[T]_{X=0} = [T]_{\mathrm{Fo}=0} = T_0$ will be

$$\Theta(X, \mathrm{Fo}) = \frac{T(X, \mathrm{Fo}) - \left(T_{\mathrm{w}} + \frac{q}{\alpha}\right)}{T_0 - \left(T_{\mathrm{w}} + \frac{q}{\alpha}\right)} = \left\{\begin{matrix} \exp(-(m+1)\,\mathrm{Bi\,Fo}), & X > \mathrm{Fo} \\ \exp(-(m+1)\,\mathrm{Bi}\,X), & X < \mathrm{Fo} \end{matrix}\right\}.$$

The upper lines for the heat conduction in a plate ($m = 0$), a cylinder ($m = 1$), and a sphere ($m = 2$) have been found singly by a different approach [19]. Integral averaging of the temperature in a fluid flow over the cross-section area enables us to propose a simplified method for solving problems of heat transfer in circular channels $\Omega(R_1 \le r \le R_2)$ at boundary conditions of the second and third kinds:

$$\frac{\partial T}{\partial t} + W(r)\frac{\partial T}{\partial z} = \frac{a}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right), \quad T(r, z, 0) = T_0,$$

$$\left(-\lambda\frac{\partial T}{\partial r}\right)_{r=R_1} = q_1(z, t), \quad \left(-\lambda\frac{\partial T}{\partial r}\right)_{r=R_2} = \alpha\left[T(R_2, z, t) - T_{\mathrm{w}}\right] + q_2(z, t), \quad q_1 > q_2,$$

where the first boundary condition is created by a fuel element in the form of a circular rod whose surface is at the same time the internal wall of the channel. Let us integrate the heat transfer equation with respect to the channel thickness

$$\frac{\partial}{\partial t}\left(\int_{R_1}^{R_2} Trdr\right) + \frac{\partial}{\partial z}\left(\int_{R_1}^{R_2} TWrdr\right) = a\left[R_2\left(\frac{\partial T}{\partial r}\right)_{r=R_2} - R_1\left(\frac{\partial T}{\partial r}\right)_{r=R_1}\right]$$

with the use of the values

$$\left(\frac{\partial T}{\partial r}\right)_{r=R_1} = -\frac{q_1(z, t)}{\lambda}, \quad \left(\frac{\partial T}{\partial r}\right)_{r=R_2} = -\frac{q_1(z, t)}{\lambda} - \frac{\alpha}{\lambda}\left[T(R_2, z, t) - T_{\mathrm{w}}\right]$$

on the assumption that the mass-average temperature coincides with the integral-average temperature. For a liquid-metal coolant with a high heat conductivity in thermally narrow channels ($\mathrm{Bi} = \frac{\alpha R}{\lambda} << 1$, $R = R_2 - R_1$) the temperature in the wall layer $T(R_2, z, t) = T(z, t)$ can be equated to the mass-average temperature. Then the problem stated for $T(z, t)$ transforms into a first-order equation, whose integration at the initial conditions $T(z, 0) = T_0$ solves the stated problem at a given heat inflow $q_1(z, t)$. Thus, when the boundary loads are expressed in terms of the temperature gradient (boundary conditions of the second and third kind), then the integral averaging over the elliptic coordinates reduces the investigation of the heat transfer in a continuum with a nonuniformly distributed parameter (temperature in space and time) to the definition of the solution in a medium with a uniform parameter over the cross-section of the channel in the form of the temperature change along the length and with time as a response of one differential inertial link.

At a plug flow ($w = 1$), in the constituent system (13) the matrixes $B = C$, $F = D$ and the elements of the response matrix $\bar{a}^*(s, p)$ with respect to the parameter $\sigma = s + p$ reduce to simpler expressions, and this makes it possible to carry out decomposition into sums of blocks from responses of two-parameter thermally inertial links. Investigations of problems of nonstationary heat transfer in channels reduce essentially to the determinations of the eigenvalues of $p_i^{(n)} = s_i^{(n)} = \sigma_i^{(n)}$ and functions for problems of heat conduction in prismatic bodies with the same convex cross-sections Ω.

Assuming in Eq. (21) $w(\xi, m) = 0$, let us describe the method for seeking a higher-accuracy solution by a one-component representation along the optimal space axis whose coordinate function depends on the kind of distribution $\psi_0(\xi)$ of the internal thermal load. For a uniform distribution, such a base coordinate will be $\psi_1(\xi, m) = \frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2$ and the solution at $f(\mathrm{Fo}) = 1$ reduces to the form

$$T(\xi, \mathrm{Fo}, \mathrm{Bi}, m) = T_0 + \frac{q_v R^2}{2\lambda (m+1)} \left(\frac{\mathrm{Bi}+2}{\mathrm{Bi}} - \xi^2 \right) \left[1 - \exp\left(- \frac{\mathrm{Bi}\,(m+1)\,(m+5)\,(\mathrm{Bi}+m+3)\,\mathrm{Fo}}{2\mathrm{Bi}^2 + 2\,(m+5)\,\mathrm{Bi} + (m+3)\,(m+5)} \right) \right]. \tag{34}$$

For the power distribution $\psi_0(\xi, \delta) = 1 + \delta\xi^n$ at $n = 4$ and the control function from the class $O(f(\mathrm{Fo})) = 1$, by solving the stationary problem with a source $q_v(1 + \delta\xi^4)$ we find

$$\psi_1(\xi, m, \mathrm{Bi}, \delta) = 6\,((m+5) + \delta\,(m+1))\,\mathrm{Bi}^{-1} + 3\,(m+5) + \delta\,(m+1) - 3\,(m+5)\,\xi^2 - \delta\,(m+1)\,\xi^6 . \tag{35}$$

At such an optimal space coordinate in the Riemann space the integrated nonstationary temperature for the three fuel elements reduces to the form

$$T(\xi, \mathrm{Fo}, m, \mathrm{Bi}, \delta) = T_0 + \frac{q_v R^2 p_1^{(1)}(m, \delta)}{6\lambda\,(m+1)\,(m+5)}\, \psi_1(\xi, m, \mathrm{Bi}, \delta) \int_0^{\mathrm{Fo}} f(\tau) \exp\left[- p_1^{(1)}(m, \delta)\,(\mathrm{Fo} - \tau) \right] d\tau , \tag{36}$$

where

$$p_1^{(1)} = A_{11}/C_{11} ; \quad A_{11} = -\int_0^1 \frac{d}{d\xi} \left(\xi^m \frac{d\psi_1}{d\xi} \right) \psi_1 d\xi > 0 ; \quad C_{11} = \int_0^1 \psi_1^2 \xi^m d\xi .$$

At $f(\mathrm{Fo}) = 1$, from this solution we obtain the expression

$$T(\xi, \mathrm{Fo}, m, \mathrm{Bi}, \delta) = T_0 + \frac{q_v R^2}{6\lambda\,(m+1)\,(m+5)}\, \psi_1(\xi, m, \mathrm{Bi}, \delta) \left[1 - \exp\left(- p_1^{(1)}(m, \delta)\,\mathrm{Fo} \right) \right], \tag{37}$$

which at $\delta = 0$ coincides with temperature (34). A special parameters δ^* determined from the condition

$$\int_0^1 \psi_0(\xi, \delta)\,\xi^m d\xi = \int_0^1 (1 + \delta\xi^n)\,\xi^m d\xi = 0\,, \quad n = 4\,, \tag{38}$$

will be $\delta^*(m) = -(m+5)/(m+1)$. Then function (35) is appreciably simplified and reduces to the form $\psi_1(\xi, m, \mathrm{Bi}, \delta^*) = (m+5)(2 - 3\xi^2 + \xi^6)$. Solutions (36), (37) with such a parameter δ^* transform into the expressions

$$T(\xi, \mathrm{Fo}, m, \delta^*) = T_0 + \frac{q_v R^2 p_1^{(1)}(m, \delta^*)}{6\lambda(m+1)}\left(2 - 3\xi^2 + \xi^6\right)\int_0^{\mathrm{Fo}} f(\tau)\exp\left[-p_1^{(1)}(m, \delta^*)(\mathrm{Fo} - \tau)\right]d\tau\,, \tag{39}$$

$$T(\xi, \mathrm{Fo}, m, \delta^*) = T_0 + \frac{q_v R^2}{6\lambda(m+1)}\left(2 - 3\xi^2 + \xi^6\right)\left[1 - \exp\left(-p_1^{(1)}(m, \delta^*)\,\mathrm{Fo}\right)\right], \tag{40}$$

where $\delta^*(0) = -5$; $\delta^*(1) = -3$; $\delta^*(2) = -7/3$; $p_1^{(1)}(0, \delta^*) = 2.799$; $p_1^{(1)}(1, \delta^*) = 6.364$; $p_1^{(1)}(2, \delta^*) = 10.662$.

At internal thermal load distributions with property (38) in the domain Ω the body surfaces become adiabatic and the process of transition to the stationary thermal state is described by formulas (39), (40). For example, inside a sphere at $\psi_0(\xi, \delta^*) = 1 - 7/\xi^4$ such stabilization of the temperature field occurs with heat release in the central part $0 \le \xi < 0.81$ and absorption in the spherical shell $0.81 \le \xi < 1$. With a special parameter δ^* the first basic coordinate function no longer depends on both the Bi number and the parameter m and is equal for all the three simple bodies. Simultaneously, it satisfies the zero boundary conditions of the first, second, and third kind in spite of the fact that the problem was stated only at boundary conditions of the third kind. Analogous solutions with such properties of the thermal state can be found at another exponent n in the function $\psi_0(\xi, \delta^*) = 1 + \delta\xi^n$, and the two-parameter representation $\psi_0(\xi, \delta_1, \delta_2) = 1 + \delta_1\xi^2 + \delta_2\xi^4$ makes it possible to obtain the effect of self-organization of the transition to the steady state of the transfer potential in a body with an adiabatic boundary at a minimum thickness of the heat absorption layer. Note that the stationary heat conduction problem at internal thermal loads of constant signs and zero boundary conditions of the second kind in any bounded domain has no solution.

If we assume $\psi_0(\xi, \delta) = \delta + \xi^2$, then the optimal coordinate function will be $\psi_1(\xi, m, \delta) = 4(\delta(m+3) + (m+1))\mathrm{Bi}^{-1} + 2\delta(m+3) + (m+1) - 2(m+3)\delta\xi^2 - (m+1)\xi^4$, and at $\delta^*(m) = -(m+1)/(m+3)$ when the heat absorption occurs already in the central part, we arrive at a more simplified basis

$$\psi_1(\xi, m, \delta^*(m)) = -(m+1)\left(\xi^4 - 2\xi^2 + 1\right).$$

The passing of the temperature to the stationary mode in the case of $f(\mathrm{Fo}) = 1$ in the three simple bodies with adiabatic surfaces is defined by the formula

$$T(\xi, \mathrm{Fo}, \delta^*(m)) = T_0 + \frac{q_v R^2}{4\lambda(m+3)}\left(\xi^4 - 2\xi^2 + 1\right)\left[1 - \exp\left(-p_1^{(1)}(\delta^*)\,\mathrm{Fo}\right)\right],$$

where $p_1^{(1)}(\delta^*(0)) = 3$; $p_1^{(1)}(\delta^*(1)) = 6.667$; $p_1^{(1)}(\delta^*(2)) = 11$.

Now, let us consider, as distributions of the local internal loading powers, any eigenfunctions of the problems of heat conduction in the three bodies, i.e., assume $\psi_0(\xi) = \psi_k(\xi, m)$, where $\psi_k(\xi, 0) = \cos(\mu_k\xi)$; $\psi_k(\xi, 1) = J_0(\mu_k\xi)$; $\psi_k(\xi, 2) = \dfrac{\sin(\mu_k\xi)}{\mu_k\xi}$, and μ_k stand for the roots of the characteristic equations at the three kinds of boundary conditions. For example, for the cylinder such equations are [9]

$$J_0(\mu) = 0\,, \quad J_1(\mu) = 0\,, \quad \frac{J_0(\mu)}{J_1(\mu)} = \frac{\mu}{\mathrm{Bi}}\,.$$

The systems approach to the realization of the resolving algorithm in a rigorous space for the internal load $q_v\psi_k(\xi, m)F(\mathrm{Fo})$ gives the exact solution

$$T(\xi, \mathrm{Fo}, m) = T_0 + \frac{q_v R^2}{\lambda} \psi_k(\xi, m) \int_0^{\mathrm{Fo}} f(\tau) \exp\left[\mu_k^2(m)(\mathrm{Fo} - \tau)\right] d\tau \tag{41}$$

and in the circular rod at a constant control function ($f(\mathrm{Fo}) = 1$) we obtain

$$T(\xi, \mathrm{Fo}, 1) = T_0 + \frac{q_v R^2}{\lambda \mu_k^2(1)} J_0(\mu_k \xi) \left[1 - \exp\left(-\mu_k^2(1)\,\mathrm{Fo}\right)\right]. \tag{42}$$

Since at boundary conditions of the second kind for all bodies to the zero eigenvalue there corresponds $\psi_1(\xi, m) = 1$, from the orthogonality of the system of eigenfunctions

$$\int_0^1 \psi_1(\xi, m)\, \psi_k(\xi, m)\, \xi^m d\xi = \int_0^1 \psi_k(\xi, m)\, \xi^m d\xi = 0\,, \quad \forall k \ge 2\,,$$

it follows that the distribution function $\psi_0(\xi) = \psi_j(\xi, m)$ satisfies condition (38) at $\forall j \ge 2$, and the exact solution (41) describes the process of transition to the stationary state at adiabatic boundaries. Note that in [14] the functions $\psi_0(\xi) = \psi_1(\xi, m)$ as source powers of fuel elements are considered only for the first eigenvalue $\mu_1^2(m)$, since for the other eigenvalues the eigenfunctions are alternating.

In conclusion, we give, without derivations, the temperature values in multidimensional domains. Inside a rectangular rod $\Omega(-h \le x \le h,\ -b \le y \le b)$ with a heat source $q_v[(1 + \beta) - (\eta^2 + \beta\xi^2)]f$ the temperatures at $f(\mathrm{Fo}) = 1$ and $f(\mathrm{Fo}) = 1 - \exp(-\mathrm{PdFo})$ are determined by the formulas

$$T(\xi, \eta, \mathrm{Fo}) = T_0 + \frac{q_v h^2}{\lambda(1 + \beta)} \left(1 - \xi^2\right)\left(1 - \eta^2\right)\left[1 - \exp\left(-p_1^{(1)}(\beta)\,\mathrm{Fo}\right)\right], \quad p_1^{(1)}(\beta) = 2.5\,(1 + \beta)\,,$$

$$T(\xi, \eta, \mathrm{Fo}) = T_0 + \frac{q_v h^2}{\lambda(1 + \beta)} \left(1 - \xi^2\right)\left(1 - \eta^2\right)\left[1 - \frac{\mathrm{Pd}\exp\left(-p_1^{(1)}\,\mathrm{Fo}\right) - p_1^{(1)}\exp(-\mathrm{Pd\,Fo})}{\mathrm{Pd} - p_1^{(1)}(\beta)}\right].$$

In a square rod ($h = b$, $\beta = 1$) at a source $q_v \cos \mu_1\xi \cos \mu_1\eta$ and heat exchange with the environment whose temperature is T_0, the expression for $T(\xi, \eta, \mathrm{Fo})$ is reduced to the form

$$T(\xi, \eta, \mathrm{Fo}) = T_0 + \frac{q_v h^2}{2\lambda\mu_1^2} \cos \mu_1\xi \cos \mu_1\eta \left[1 - \exp\left(-\mu_1^2 \mathrm{Fo}\right)\right],$$

where μ_1 is the first root of the equation $\cot \mu = \mu/\mathrm{Bi}$, $\mathrm{Bi} = \alpha h/\lambda$, $h = b$. Inside a prism with adiabatic faces and alternating load $q_v[(1 + \beta) - 3(\xi^2 + \beta\eta^2)]$ the temperature field is given by the formula

$$T(\xi, \eta, \mathrm{Fo}) = T_0 + \frac{q_v h^2}{4\lambda} \left(\xi^4 - 2\xi^2 + \eta^4 - 2\eta^2 + \frac{14}{15}\right)\left(1 - \exp(-5(1 + \beta)\,\mathrm{Fo})\right).$$

In an isosceles triangular prism $\Omega\left(y \le \frac{b}{h}x,\ y \ge -\frac{b}{h}x,\ 0 \le x \le h\right)$ with $q_v[(3\xi - 1) + \beta(1 - \xi)]$ the temperature at constant boundary conditions of the first kind is reduced to the form

$$T(\xi, \eta, \mathrm{Fo}, \beta) = T_0 + \frac{3q_v h^2}{2\lambda (3+\beta)} \left(\xi^2 - \eta^2\right)(1-\xi)\left[1 - \exp\left(-7(\beta+3)\,\mathrm{Fo}\right)\right]$$

and for a prism with three equal faces ($\beta = 3$), the steady source $2q_v$ creates the temperature change

$$T(\xi, \eta, \mathrm{Fo}, \beta) = T_0 + \frac{q_v h^2}{4\lambda} \left(\xi^2 - \eta^2\right)(1-\xi)\left[1 - \exp(-42\mathrm{Fo})\right],$$

where for comparison of the stabilization rate to the stabilizations of the temperatures in the three simple bodies one should relate Fo to the equivalent radius $R = h/3$; then instead of 42 we will obtain $p_1^{(1)} = 4.667$.

At a linear increase in the wall temperature without internal loading we find

$$T(\xi, \eta, \mathrm{Fo}, 3) = T_0 + \Delta T\mathrm{Fo} - 0.25\Delta T\left(\xi^2 - \eta^2\right)(1-\xi)\left[1 - \exp(-42\mathrm{Fo})\right].$$

Inside a paraboloid of revolution $\Omega\left(x \geq \frac{h}{b^2}(y^2 + z^2),\ 0 \leq x \leq h\right)$, the temperature as a response to the source $q_v[1+2\beta(1-\xi)]f(\mathrm{Fo})$ at stationary loads ($f = 1$) is determined by the formula

$$T(\xi, \eta, \zeta, \mathrm{Fo}, \beta) = T_0 + \frac{q_v h^2}{2\lambda}\left(\xi - \eta^2 - \zeta^2\right)(1-\xi)\left[1 - \exp\left(-2.25(5+4\beta)\,\mathrm{Fo}\right)\right].$$

CONCLUSIONS

1. The simultaneous use of the positive properties of the integral transforms and the orthogonalprojection of the residual, when the entire domain of variability of elliptic coordinates is taken as a finite element, has enabled us to develop an effective method for defining analytical solutions of nonstationary problems of heat transfer in channels of complex profiles and heat conduction in multidimensional bodies.

2. The universality of the method and the possibility of solving a wide range of problems follow from the fact that the algorithm is realized at any given velocity distribution function of laminar or turbulent flows. With the introduction of the residual projection the practical need for preliminary search for a system of eigenfunctions at concrete values of W and ε for investigating heat-transfer problems by the rigorous methods of mathematical physics has disappeared. The explicit form of the Sturm–Liouville functions is obtained for a rather narrow range of problems, and for multidimensional regions it is impossible to determine the eigenfunctions and eigenvalues. The systems application of the residual projection in the set of representations of temperature fields in alternative spaces suffices for finding solutions in the second or third approximations. From the main spectrum of such solutions the first eigenfunction and the eigenvalue are obtained with a high accuracy without investigating the spectral boundary-value Sturm–Liouville problem.

3. In defining solutions in the second approximation of the problems of heat conduction in the three simple bodies with boundary conditions of the third kind, we have found formulas for calculating the first eigenvalues from which we find the dependences of the first roots of the characteristic equations for the plate, the cylinder, and the sphere throughout the range of change in the Bi number ($0 < \mathrm{Bi} < \infty$) coinciding up to the fourth digit of the decimal fraction with the exact values.

4. The determination of the sought function by arguments with an infinite interval of their change is carried out for continua, and the temperature field is found in the integral representation with infinite limits of integration. A change in the elliptic coordinates on a finite interval leads to the synthesis of the temperature by discrete spectra in the form of a sum of a series of terms, which confirms the expediency of using the exact methods of mathematical physics at unlimited intervals and the approximate numerical or analytic solution methods when the interval of change is finite. Among the rigorous methods for solving boundary-value problems in unlimited intervals are the bilateral Fourier transform and the unilateral transforms with kernels from the sine and cosine functions, and for equations in cylindrical coordinates — the integral Hankel transform with a Hankel of Bessel functions. If we introduce the defini-

tion cos $\mu\xi$ and sin $\mu = 0$ as trigonometric functions of the zero and first order, as is common for Bessel functions, then their similarity is revealed in considering the base coordinates for solving boundary-value problems in the rectangular and cylindrical coordinate systems [11]. For example, in problems for a plate and a cylinder analogous characteristic equations

$$\cos \mu = 0 , \quad \sin \mu = 0 , \quad \frac{\cos \mu}{\sin \mu} = \frac{\mu}{\mathrm{Bi}} , \quad J_0 (\mu) = 0 , \quad J_1 (\mu) = 0 , \quad \frac{J_0 (\mu)}{J_1 (\mu)} = \frac{\mu}{\mathrm{Bi}}$$

are written.

5. The uniqueness of the method in the solutions of transfer equations with variable coefficients has made it possible to investigate the problems of heat transfer in flows of Newtonian fluids with a power rheological law and of structure-viscous media with other flow velocities, and in the Hartmann MHD flow.

6. The search for a solution by the rigorous methods of mathematical physics leads to the determination of temperature fields by the system of eigenfunctions which are expressed by trigonometric, cylindrical, and other special functions. These functions are determined independent of the conditions of heat loads; therefore, for some input functions the solutions are reduced to slowly converging series, which is due to the additional difficulties in using such temperature fields in thermophysical calculations. The application of the method of orthogonal projection in alternative spaces with power coordinate functions avoids these difficulties.

7. Depending on the geometry of the body and the internal source heat loads, a method for choosing the optimal basis in the Riemannian manifold, along whose axis the exact solution or a higher-accuracy solution are found, has been proposed. In the case of a nonstationary temperature field, a simple search method for a family of isothermal surfaces, on each of which the temperature remains equal and varies only with time, has been found [8]. A special condition of internal thermal loading at which in a bounded region the self-organizing process of transition of the transfer potential to the steady state (stagnation) proceeds has been discovered. In heat conduction problems with boundary conditions of the first and third kind at such internal loading, heat exchange with the environment terminates and change-over to a problem with zero boundary conditions of the second kind occurs.

NOTATION

a, thermal diffusivity; f_0, f_1, initial distribution function of the temperature and its time derivative; t, time; T, temperature; T_0, T_{w}, constant initial temperature of the wall (medium); $w = W/w_0$, relative flow velocity of the coolant; w_0, mean velocity; λ, heat conductivity; μ, dynamic viscosity; μ_k^2, eigenvalues; ψ_0, stationary distribution of the internal heat load; Bi, Fo, Nu, Pe, Po, Pd, Biot, Fourier, Nusselt, Peclet, Pomerantsev, and Predvoditelev numbers.

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